@drvinceknight
vincent-knight.com/Talks

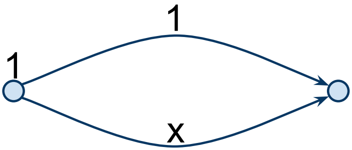
$$\begin{pmatrix} (3,2) & (1,1) \\ (0,0) & (2,3) \end{pmatrix}$$

Sagemath

4 problems

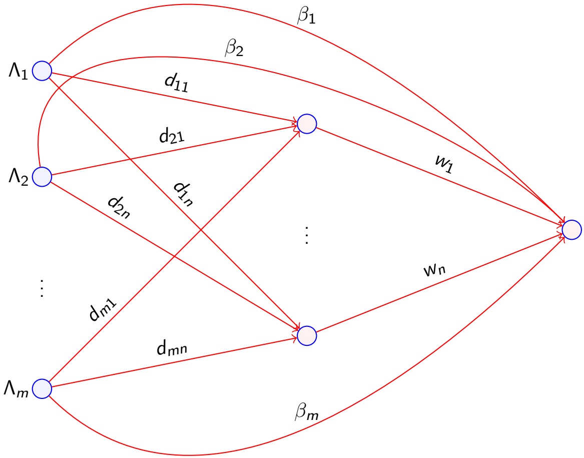
Users

- $k = 1$
- $\mathcal{P}_1 = \{1, 2\}$
- $c_1 = 1$ and $c_2 = x$
- $r = 1$



The Nash flow minimises:

$$\Phi(y, 1-y) = \sum_{e=1}^{2} \int_0^{f_e} c_e(x)dx = \int_0^{y} 1dx + \int_0^{1-y} xdx$$
$$= y + \frac{(1-y)^2}{2} = \frac{1}{2} + \frac{y^2}{2}$$
$$\Rightarrow \tilde{f} = (0, 1)$$



Λ_1
Λ_2
Λ_m
β_1
β_2
β_m
d_{11}
d_{21}
d_{1n}
d_{2n}
d_{m1}
d_{mn}
w_1
w_n

Theorem Assuming $\sum_{i=1}^{m} \Lambda_i < \sum_{j=1}^{n} c_j \mu_j$ we have:

$$\lim_{\beta_i \to \infty} PoA(\beta) < \infty \text{ for all } i \in [m]$$

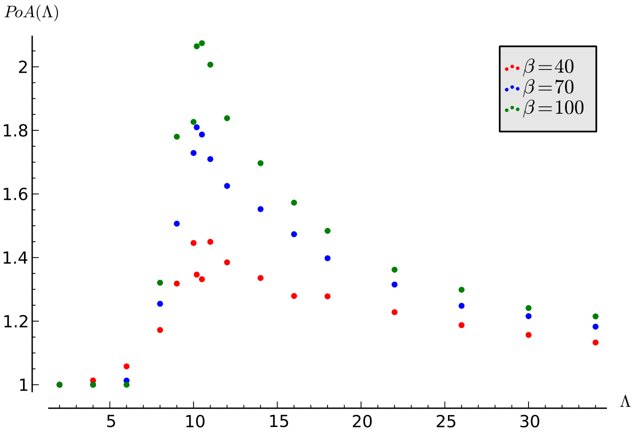
The price of anarchy increases with worth of service, up to a point.

Proof.

- $\lim_{\beta_i \to \infty} \lambda^* = k^*$ and $\lim_{\beta_i \to \infty} \tilde{\lambda} = \tilde{k}$
- As $\beta_i \to \infty$:

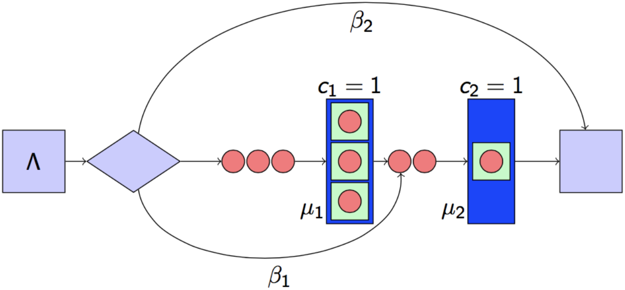
$$\sum_{i=1}^{m} \Lambda_i = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^* = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{\lambda}_{ij}$$

- $PoA(\beta) < \infty$



$PoA(\Lambda)$
2
1.8
1.6
1.4
1.2
1
5
10
15
20
25
30
Λ
$\beta = 40$
$\beta = 70$
$\beta = 100$

Price of Anarchy in Public Services *EJORS*, 2013.



β_2
$c_1 = 1$
$c_2 = 1$
Λ
μ_1
μ_2
β_1

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

Cost of a queue:

$$\epsilon(n) = \begin{cases} 1/\mu, & n < c \\ \frac{1+n}{c\mu}, & \text{otherwise} \end{cases}$$

Cost of a state when using T.

$$C_t(i,j) = \begin{cases} \epsilon_1(i) + \epsilon_2(j), t = 0 \\ \beta_1 + \epsilon_2(j), t = 1 \\ \beta_2, t = 2 \end{cases}$$

Cost of a queue:

$$\epsilon(n) = \begin{cases} 1/\mu, & n < c \\ \frac{1+n}{c\mu}, & \text{otherwise} \end{cases}$$

Cost of a state when using T.

$$C_t(i,j) = \begin{cases} \epsilon_1(i) + ?, t = 0 \\ \beta_1 + \epsilon_2(j), t = 1 \\ \beta_2, t = 2 \end{cases}$$

A Nash policy $\tilde{T}$ is a solution to:

$$\min_{t \in \{0,1,2\}} C_t(i,j) = C_{T_{ij}}(i,j) \text{ for all } i,j$$

For example $\mu = (3, 1)$, $c = (4, 2)$, $\beta = (.55, 4)$ and $\Lambda = 2$:

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

$$T^* = \begin{pmatrix} 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

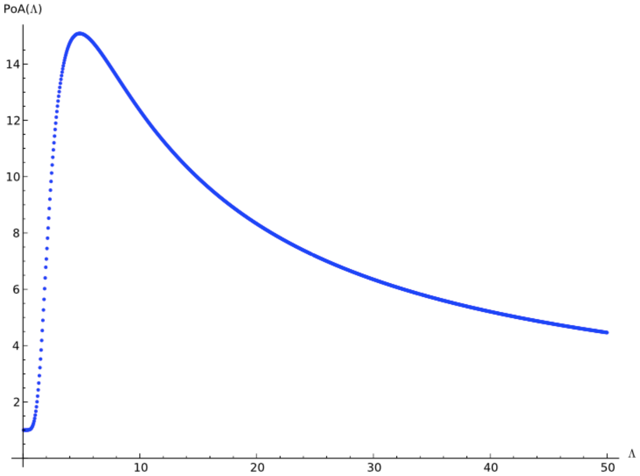
For example $\mu = (3, 1)$, $c = (4, 2)$, $\beta = (.55, 4)$ and $\Lambda = 2$:

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

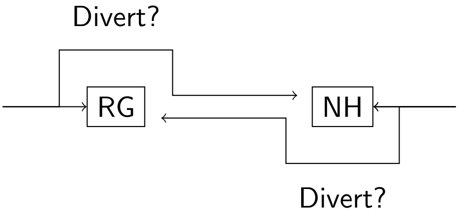
$$T^* = \begin{pmatrix} 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

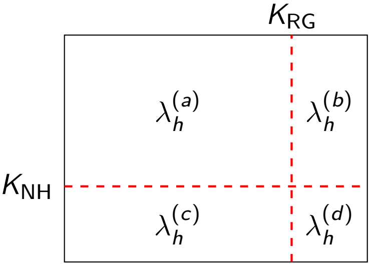
$C(\tilde{T}) = 1.9937$, $C(T^*) = .2818$

PoA=7.0749

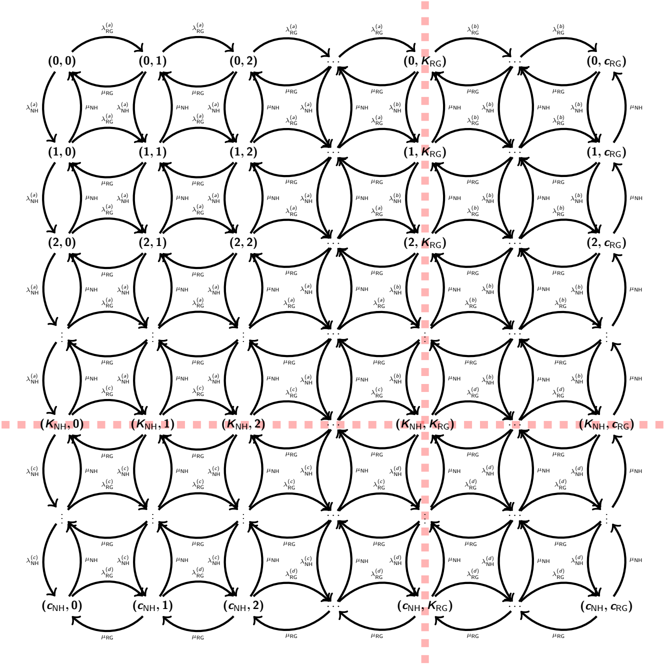

PoA(Λ)
14
12
10
8
6
4
2
10
20
30
40
50
Λ

What about the controllers?

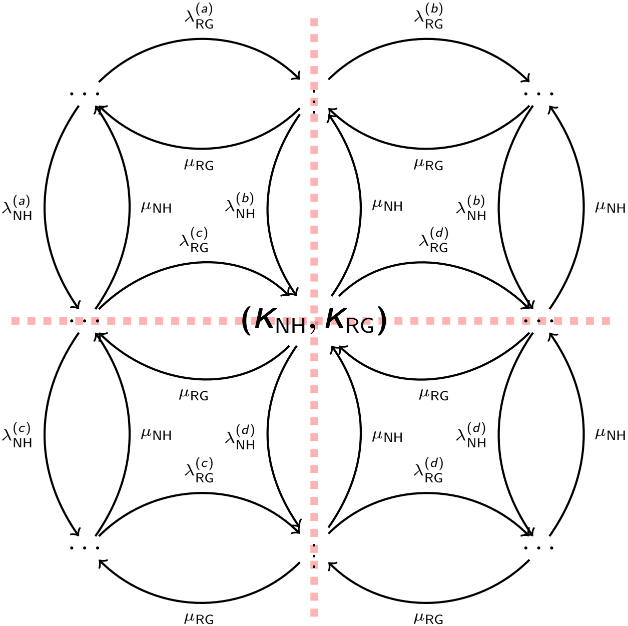

Divert?
RG
NH
Divert?



K_{RG}
K_{NH}
$\lambda_h^{(a)}$
$\lambda_h^{(b)}$
$\lambda_h^{(c)}$
$\lambda_h^{(d)}$



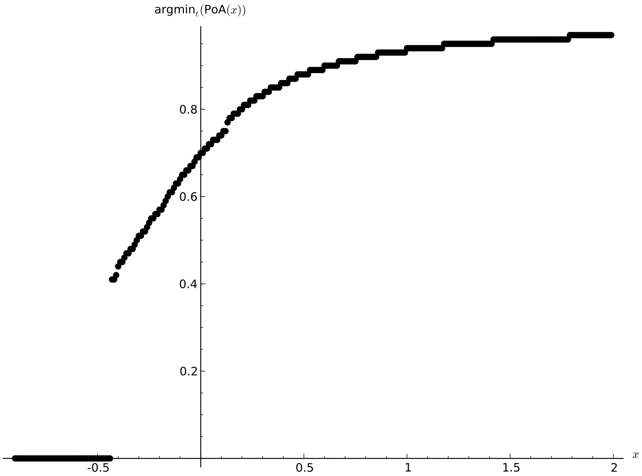
(0, 0)
(0, 1)
(0, 2)
(0, K_{RG})
(0, c_{RG})
(1, 0)
(1, 1)
(1, 2)
(1, K_{RG})
(1, c_{RG})
(2, 0)
(2, 1)
(2, 2)
(2, K_{RG})
(2, c_{RG})
(K_{NH}, 0)
(K_{NH}, 1)
(K_{NH}, 2)
(K_{NH}, K_{RG})
(K_{NH}, c_{RG})
(c_{NH}, 0)
(c_{NH}, 1)
(c_{NH}, 2)
(c_{NH}, K_{RG})
(c_{NH}, c_{RG})
$\lambda_{RG}^{(a)}$
$\lambda_{RG}^{(b)}$
$\lambda_{RG}^{(c)}$
$\lambda_{RG}^{(d)}$
$\lambda_{NH}^{(a)}$
$\lambda_{NH}^{(b)}$
$\lambda_{NH}^{(c)}$
$\lambda_{NH}^{(d)}$
μ_{RG}
μ_{NH}


$\lambda_{RG}^{(a)}$
$\lambda_{RG}^{(b)}$
μ_{RG}
μ_{RG}
$\lambda_{NH}^{(a)}$
μ_{NH}
$\lambda_{NH}^{(b)}$
μ_{NH}
$\lambda_{NH}^{(b)}$
μ_{NH}
$\lambda_{RG}^{(c)}$
$\lambda_{RG}^{(d)}$
(K_{NH}, K_{RG})
μ_{RG}
μ_{RG}
$\lambda_{NH}^{(c)}$
μ_{NH}
$\lambda_{NH}^{(d)}$
μ_{NH}
$\lambda_{NH}^{(d)}$
μ_{NH}
$\lambda_{RG}^{(c)}$
$\lambda_{RG}^{(d)}$
μ_{RG}
μ_{RG}

$$A = \begin{pmatrix} (U_{\mathrm{NH}}(1,1) - t)^2 & \dots & (U_{\mathrm{NH}}(1, c_{\mathrm{RG}}) - t)^2 \\ (U_{\mathrm{NH}}(2,1) - t)^2 & \dots & (U_{\mathrm{NH}}(2, c_{\mathrm{RG}}) - t)^2 \\ \vdots & \ddots & \vdots \\ (U_{\mathrm{NH}}(c_{\mathrm{NH}},1) - t)^2 & \dots & (U_{\mathrm{NH}}(c_{\mathrm{NH}}, c_{\mathrm{RG}}) - t)^2 \end{pmatrix}$$

$$B = \begin{pmatrix} (U_{\mathrm{RG}}(1,1) - t)^2 & \dots & (U_{\mathrm{RG}}(1, c_{\mathrm{RG}}) - t)^2 \\ (U_{\mathrm{RG}}(2,1) - t)^2 & \dots & (U_{\mathrm{RG}}(2, c_{\mathrm{RG}}) - t)^2 \\ \vdots & \ddots & \vdots \\ (U_{\mathrm{RG}}(c_{\mathrm{RG}},1) - t)^2 & \dots & (U_{\mathrm{RG}}(c_{\mathrm{RG}}, c_{\mathrm{RG}}) - t)^2 \end{pmatrix}$$

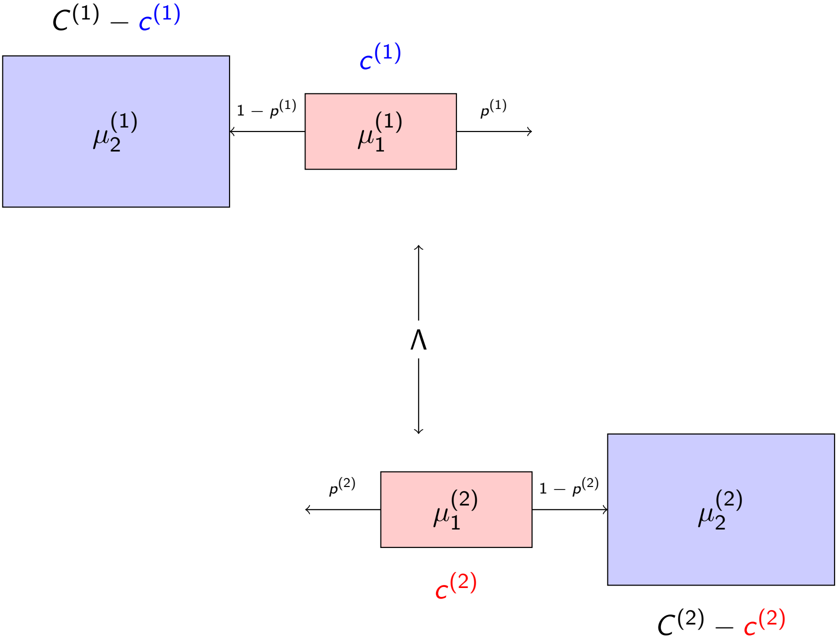
Theorem.
Let $f_h(k) : [1, c_{\bar{h}}] \rightarrow [1, c_h]$ be the best response of player $h \in \{\mathrm{NH}, \mathrm{RG}\}$ to the diversion threshold of $\bar{h} \neq h$ ($\bar{h} \in \{\mathrm{NH}, \mathrm{RG}\}$). If $f_h(k)$ is a non-decreasing function in k then the game has at least one Nash Equilibrium in Pure Strategies.



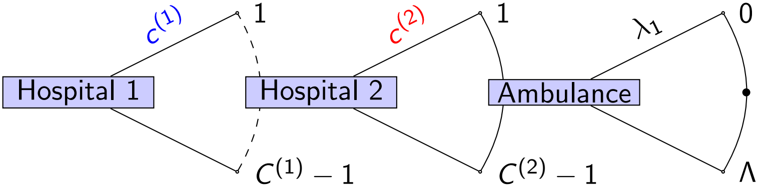
argmin$_t$(PoA(x))
0.8
0.6
0.4
0.2
-0.5
0.5
1
1.5
2
x

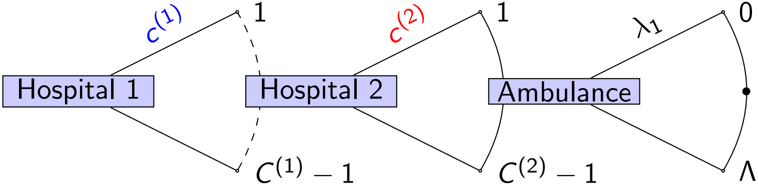
Measuring the Price of Anarchy in Critical Care Unit Interactions, *Submitted to JORS*


UNED ACHOSION BRYS
EMERGENCY UNIT
AMBULANCE
EMERGENCY AMBULANCE
AMBIWLANS
AMBULANCE
AMBULANCE



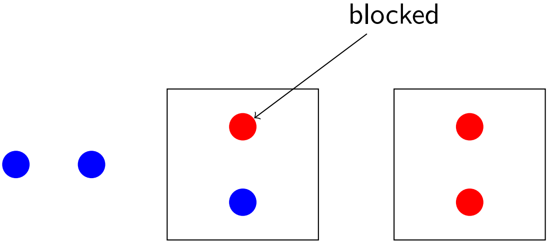
$C^{(1)} - c^{(1)}$
$c^{(1)}$
$\mu_2^{(1)}$
$1 - p^{(1)}$
$\mu_1^{(1)}$
$p^{(1)}$
Λ
$p^{(2)}$
$\mu_1^{(2)}$
$1 - p^{(2)}$
$\mu_2^{(2)}$
$c^{(2)}$
$C^{(2)} - c^{(2)}$


$c^{(1)}$
1
Hospital 1
$C^{(1)} - 1$
$c^{(2)}$
1
Hospital 2
$C^{(2)} - 1$
λ_1
0
Ambulance
Λ

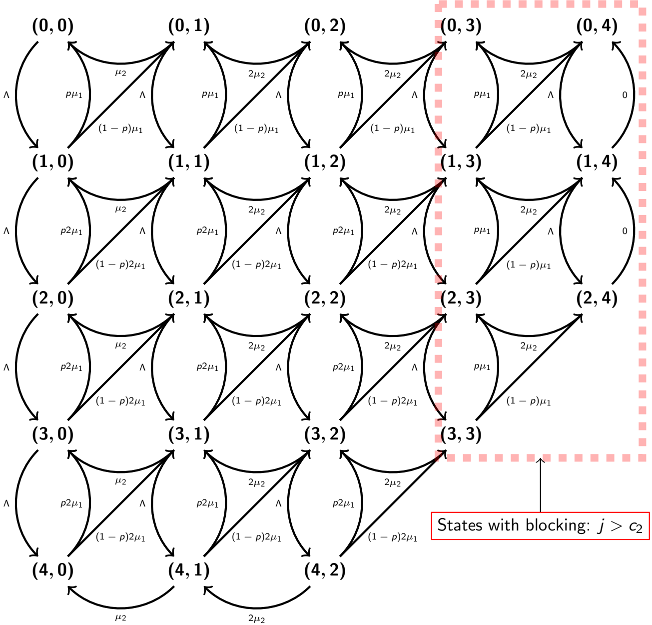


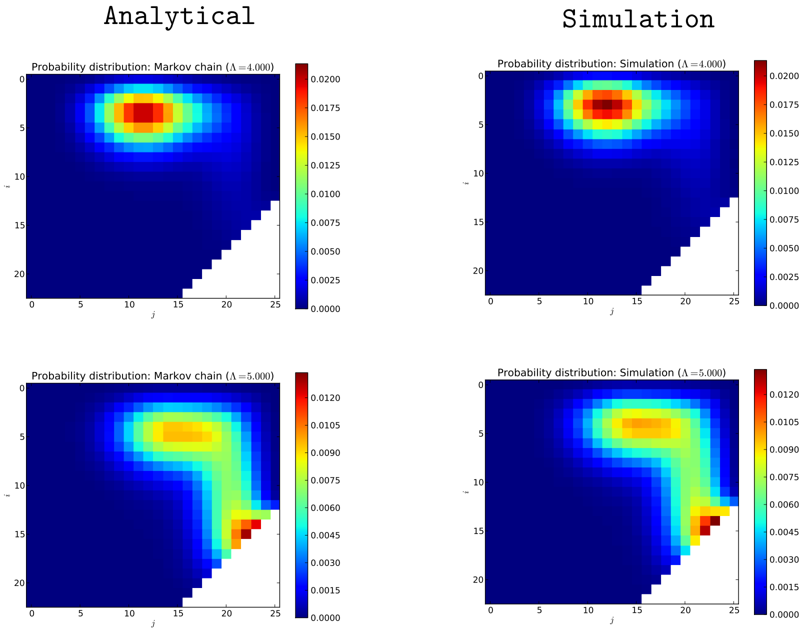
$$\left(|u_1^{(1)} - u_2^{(1)}|, |u_1^{(2)} - u_2^{(2)}|, |w^{(1)} - w^{(2)}|\right)$$

$$S = \{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid 0 \leq j \leq c_1 + c_2,\ 0 \leq i \leq c_1 + N - \max(j - c_2, 0)\}$$



$$q_{(i_1,j_1),(i_2,j_2)} = \begin{cases} \Lambda, & \text{if } \delta = (1,0) \\ \min(c_1 - \max(j_1 - c_2, 0), i_1)(1-p)\mu_1, & \text{if } \delta = (-1,1) \\ \min(c_1 - \max(j_1 - c_2, 0), i_1)p\mu_1, & \text{if } \delta = (-1,0) \\ \min(c_2, j_1)\mu_2, & \text{if } \delta = (0,-1) \end{cases}$$

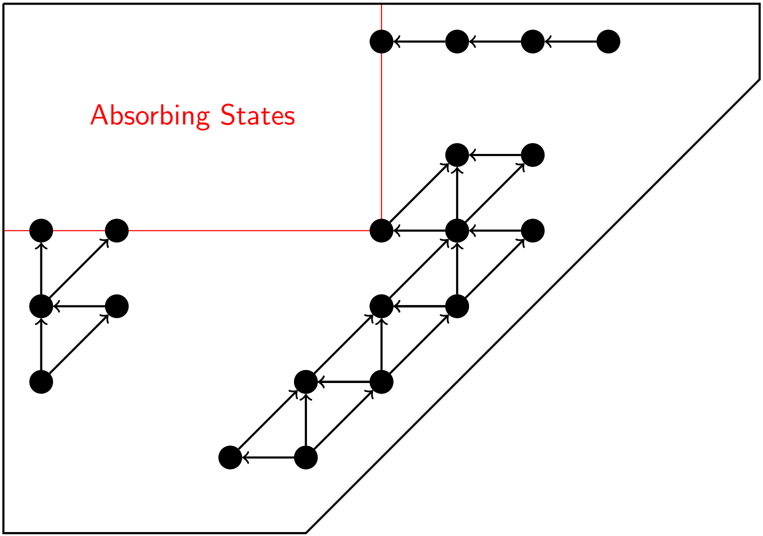

(0, 0)
(0, 1)
(0, 2)
(0, 3)
(0, 4)
(1, 0)
(1, 1)
(1, 2)
(1, 3)
(1, 4)
(2, 0)
(2, 1)
(2, 2)
(2, 3)
(2, 4)
(3, 0)
(3, 1)
(3, 2)
(3, 3)
(4, 0)
(4, 1)
(4, 2)
Λ
μ_2
$2\mu_2$
$p\mu_1$
$p2\mu_1$
$(1-p)\mu_1$
$(1-p)2\mu_1$
0
States with blocking: $j > c_2$



Analytical
Simulation
Probability distribution: Markov chain ($\Lambda = 4.000$)
Probability distribution: Simulation ($\Lambda = 4.000$)
Probability distribution: Markov chain ($\Lambda = 5.000$)
Probability distribution: Simulation ($\Lambda = 5.000$)
i
j
0
5
10
15
20
25
0.0000
0.0025
0.0050
0.0075
0.0100
0.0125
0.0150
0.0175
0.0200
0.0015
0.0030
0.0045
0.0060
0.0075
0.0090
0.0105
0.0120

Expected wait:

$$w = \frac{\sum_{(i,j)\in S_A} c(i,j)\pi_{(i,j)}}{\sum_{(i,j)\in S_A} \pi(i,j)}$$

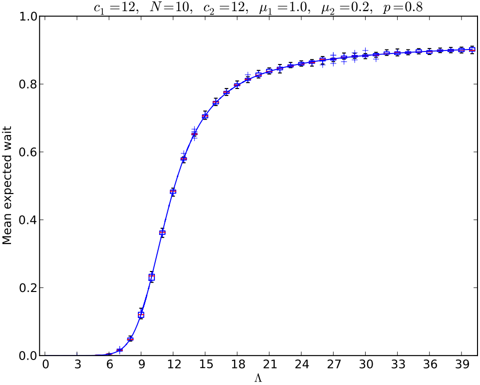


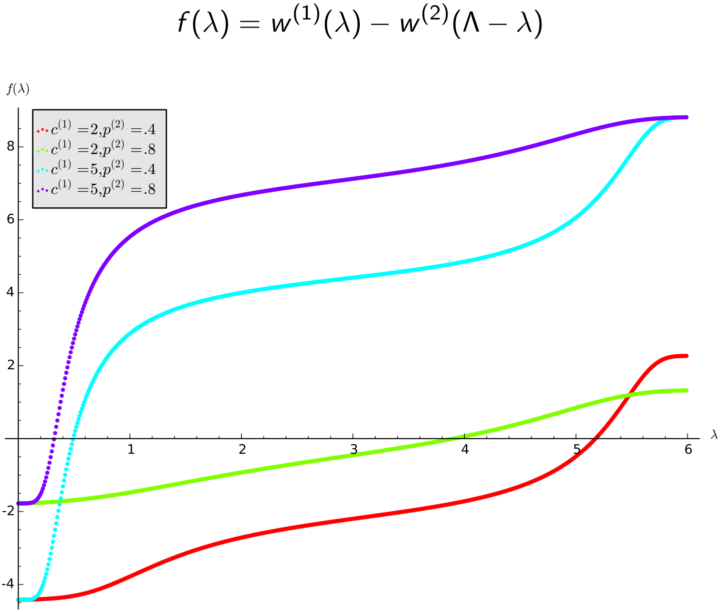
Sojourn time in state (i, j):

$$w(i, j) = \frac{1}{\min(c_2, j)\mu_2 + \min(c_1 - \max(j - c_2, 0), i)\mu_1}$$

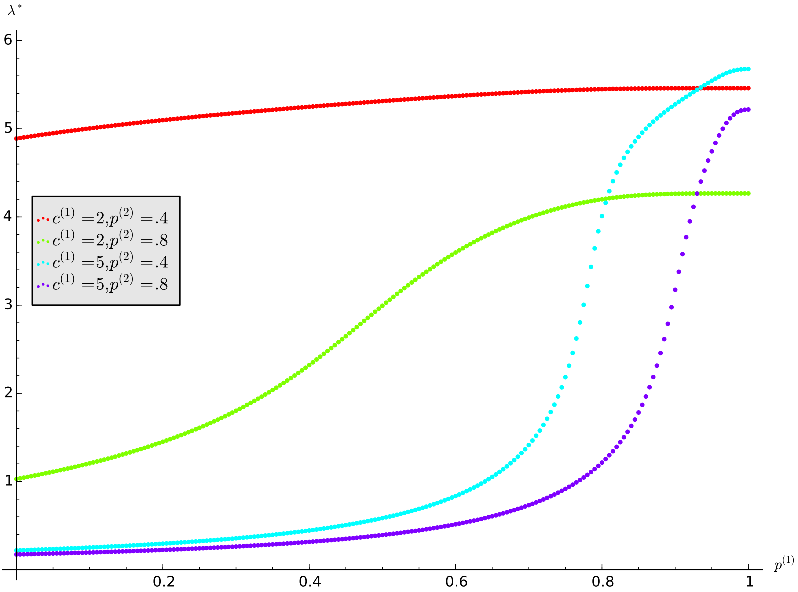
Cost of state (i, j):

$$c(i, j) = \begin{cases} 0, & \text{if } (i, j) \in A \\ w(i, j) + p_{s_2} c(i, j - 1) + p_{s_1}(pc(i - 1, j) + (1 - p)c(i - 1, j + 1)), & \text{otherwise} \end{cases}$$





$f(\lambda) = w^{(1)}(\lambda) - w^{(2)}(\Lambda - \lambda)$
$f(\lambda)$
$c^{(1)} = 2, p^{(2)} = .4$
$c^{(1)} = 2, p^{(2)} = .8$
$c^{(1)} = 5, p^{(2)} = .4$
$c^{(1)} = 5, p^{(2)} = .8$
8
6
4
2
-2
-4
1
2
3
4
5
6
λ



λ^*
6
5
4
3
2
1
0.2
0.4
0.6
0.8
1
$p^{(1)}$
$c^{(1)} = 2, p^{(2)} = .4$
$c^{(1)} = 2, p^{(2)} = .8$
$c^{(1)} = 5, p^{(2)} = .4$
$c^{(1)} = 5, p^{(2)} = .8$

$$\Lambda = 6,\, C^{(1)} = 6,\, C^{(2)} = 4,\, N^{(1)} = N^{(2)} = 3$$

$$A = \begin{pmatrix} 0.795 & 0.688 & 0.792 \\ 0.506 & 0.488 & 0.503 \\ 0.183 & 0.159 & 0.178 \\ 0.0104 & 0.0193 & 0.00523 \\ 0.0121 & 0.108 & 0.0159 \end{pmatrix} B = \begin{pmatrix} 0.667 & 0.243 & 0.00105 \\ 0.480 & 0.154 & 0.196 \\ 0.396 & 0.0774 & 0.253 \\ 0.470 & 0.140 & 0.205 \\ 0.664 & 0.239 & 0.00837 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 3.17 & 1.18 & 2.88 \\ 5.18 & 3.90 & 4.87 \\ 5.37 & 4.39 & 5.07 \\ 5.21 & 4.01 & 4.90 \\ 3.46 & 1.67 & 3.18 \end{pmatrix} S = \begin{pmatrix} 0.672 & 0.481 & 0.672 \\ 0.381 & 0.427 & 0.429 \\ 0.315 & 0.341 & 0.352 \\ 0.376 & 0.418 & 0.423 \\ 0.666 & 0.535 & 0.671 \end{pmatrix}$$

$$\Lambda = 6,\, C^{(1)} = 6,\, C^{(2)} = 4,\, N^{(1)} = N^{(2)} = 3$$

$$A = \begin{pmatrix} 0.795 & 0.688 & 0.792 \\ 0.506 & 0.488 & 0.503 \\ 0.183 & 0.159 & 0.178 \\ 0.0104 & 0.0193 & 0.00523 \\ 0.0121 & 0.108 & 0.0159 \end{pmatrix} B = \begin{pmatrix} 0.667 & 0.243 & 0.00105 \\ 0.480 & 0.154 & 0.196 \\ 0.396 & 0.0774 & 0.253 \\ 0.470 & 0.140 & 0.205 \\ 0.664 & 0.239 & 0.00837 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 3.17 & 1.18 & 2.88 \\ 5.18 & 3.90 & 4.87 \\ 5.37 & 4.39 & 5.07 \\ 5.21 & 4.01 & 4.90 \\ 3.46 & 1.67 & 3.18 \end{pmatrix} S = \begin{pmatrix} 0.672 & 0.481 & 0.672 \\ 0.381 & 0.427 & 0.429 \\ 0.315 & 0.341 & 0.352 \\ 0.376 & 0.418 & 0.423 \\ 0.666 & 0.535 & 0.671 \end{pmatrix}$$

$\tilde{c}_1 = 4, \tilde{c}_2 = 2$ and $c_1^* = 3, c_2^* = 1$ for PoA $= 1.330$.

$$\text{PoA} = \frac{\tilde{c}_1 S \tilde{c}_2}{c_1^* S c_2^*} = \frac{\tilde{c}_1 S \tilde{c}_2}{\min S}$$

from A, B

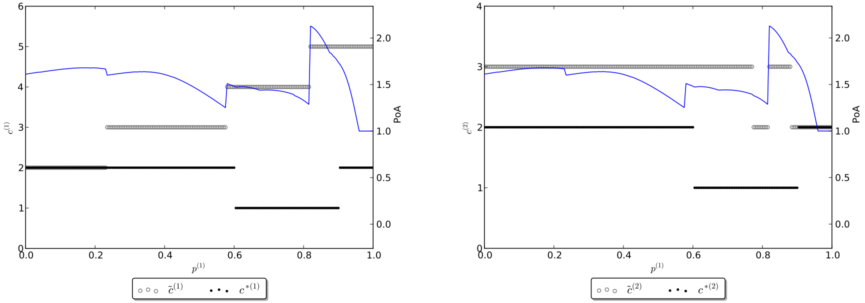
$$\text{PoA} = \frac{\tilde{c}_1 S \tilde{c}_2}{c_1^* S c_2^*} = \frac{\tilde{c}_1 S \tilde{c}_2}{\min S}$$

from A, B

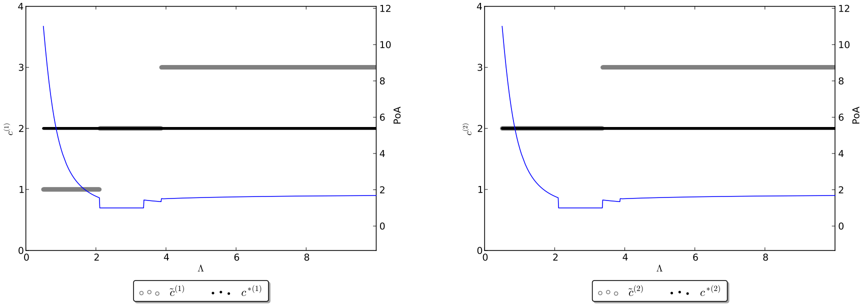
$$\text{PoA} = \frac{\tilde{c}_1 S \tilde{c}_2}{c_1^* S c_2^*} = \frac{\tilde{c}_1 S \tilde{c}_2}{\min S}$$

from $f(\lambda)$

Effect of $p^{(1)}$

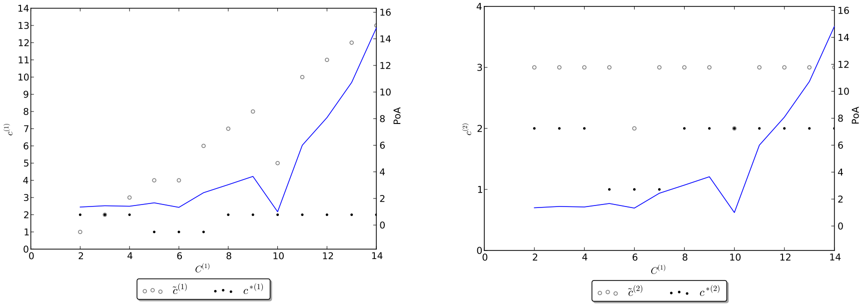


Effect of Λ

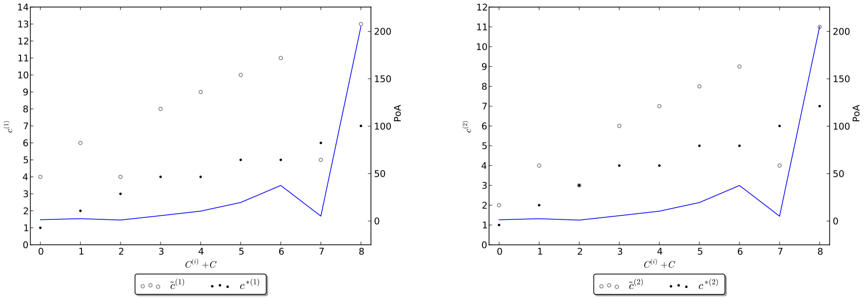


4
3
2
1
0
$c^{(1)}$
0
2
4
6
8
Λ
12
10
8
6
4
2
0
PoA
$\tilde{c}^{(1)}$
$c^{*(1)}$
4
3
2
1
0
$c^{(2)}$
0
2
4
6
8
Λ
12
10
8
6
4
2
0
PoA
$\tilde{c}^{(2)}$
$c^{*(2)}$

Effect of $C^{(1)}$



Effect of $C^{(i)}$



- A lot of potential for Game Theory + Stochastic modelling applied to Game Theory;
- Ability to model Patient + Controller behaviour;
- Potential advances for theoretical + applied contributions.

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